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Loop Quantum Cosmology, Boundary Proposals, and Inflation

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Abstract

Loop quantum cosmology of the closed isotropic model is studied with a special emphasis on a comparison with traditional results obtained in the Wheeler–DeWitt approach. This includes the relation of the dynamical initial conditions with boundary conditions such as the no-boundary or the tunneling proposal and a discussion of inflation from quantum cosmology.

1 Introduction

Traditionally, quantum cosmology has been studied in simple models which have been obtained by a classical reduction of general relativity to a system of finitely many degrees of freedom and a subsequent application of quantum mechanical methods [1, 2]. A corresponding full quantum theory of gravity, let alone a relation of the models to it, has remained unknown beyond a purely formal level. The most important question for quantum cosmology, whether or not classical singularities are absent, has not been answered positively in this approach. Instead, the classical singularity has been removed by hand and substituted with some proposals of intuitive initial conditions [1, 3, 4].

The situation has changed, however, with the advent of quantum geometry (also called loop quantum gravity, see [5, 6, 7]), a consistent canonical quantization of full general relativity. It is possible to derive cosmological models in a way analogous to the full theory [8] yielding loop quantum cosmology [9, 10]. In this reduction, the main departure from the traditional approach is the prediction that quantum Riemannian geometry has a discrete structure. The effects caused by the discreteness are most important at small volume implying that the structure close to the classical singularity is very different from that of a Wheeler–DeWitt quantization which fails to resolve the singularity problem in this

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regime. At large volume, however, the discreteness leads only to small corrections and the traditional approach is reproduced as an approximation.

Therefore, the main new results of loop quantum cosmology affect the behavior at small volume: There is no singularity [11], initial conditions for the wave function of a universe follow from the evolution equation [12], and the approach to the classical singularity is modified by non-perturbative effects which imply an inflationary period [13]. From the point of view of initial conditions, the situation looks closest to DeWitt's original proposal presenting, however, a well-defined generalization to other models [14]. So far all of these results have been derived and studied in detail in the flat isotropic model only because a non-vanishing intrinsic curvature leads to effects which complicate the loop quantization. Recently, the methods of loop quantum cosmology have been extended to homogeneous models with non-vanishing intrinsic curvature [15] such that now also a consistent quantization of the closed isotropic model is available.

Since the traditional results have all been derived for the closed model only, it is more important to compare with the effects of loop quantum cosmology for that model. The general qualitative results mentioned above remain true for the closed model, but some of them can be different quantitatively. For instance, we will see that the dynamical initial conditions resemble the no-boundary proposal more closely than the tunneling proposal. This observation might appear negative because the no-boundary proposal is widely perceived as being unsupportive to large initial values of an inflaton field needed for a sufficient amount of inflation. Here, however, we have an alternative inflationary scenario based on the modified approach to classical singularities which is also realized in the closed model. But due to the intrinsic curvature the inflationary period does not necessarily extend to arbitrarily small values of the scale factor which could lead to insufficient inflation. As a new possibility we will therefore look at the closed isotropic model as embedded in the anisotropic Bianchi IX model. This leads to additional quantum modifications of intrinsic curvature terms and of the classical behavior, implying a second period of inflation at very small volume. With this more general quantization the inflationary picture of the closed model is more complicated but comparable to that of the flat model.

2 Loop Quantum Cosmology of the Closed Isotropic Model

Quantum geometry and loop quantum cosmology are based on connection and densitized triad variables [16, 17] which in the isotropic case [10, 18] are given by $c = \Gamma - \gamma K$ and p, respectively, where K is the extrinsic curvature and the parameter Γ represents the intrinsic curvature with values $\Gamma = 0, \frac{1}{2}$ for the flat model and closed model respectively. Both variables are canonically conjugate: $\{c, p\} = \frac{1}{3}\gamma\kappa$ where $\kappa = 8\pi G$ is the gravitational constant and γ is the Barbero–Immirzi parameter [17, 19] which is a positive real number and labels classically equivalent formulations. The relation to the better known ADM variables is given by $K = -\frac{1}{2}\dot{a}$ (extrinsic curvature) and $|p| = a^2$ where a is the scale

factor. Since a triad can have two different orientations, its isotropic component p can take both signs.

The dynamics is governed by the Hamiltonian constraint [10]

$$\begin{aligned} H &= -6\gamma^{-2}\kappa^{-1}\left((c-2\Gamma)c+(1+\gamma^2)\Gamma^2\right)\mathrm{sgn}(p)\sqrt{|p|} & (1)\\ &= \left\{\begin{array}{ll} -6\gamma^{-2}\kappa^{-1}c^2\,\mathrm{sgn}(p)\sqrt{|p|} & \text{for the flat model}\\ -6\gamma^{-2}\kappa^{-1}(c^2-c+\frac{1}{4}(1+\gamma^2))\,\mathrm{sgn}(p)\sqrt{|p|} & \text{for the closed model}\end{array}\right. & (2)\end{aligned}$$

or in terms of the extrinsic curvature

$$\begin{aligned} H &= -6\kappa^{-1}(K^2+\Gamma^2)\,\mathrm{sgn}(p)\sqrt{|p|}\\ &= \left\{\begin{array}{ll} -\frac{3}{2}\kappa^{-1}\dot{a}^2a & \text{flat}\\ -\frac{3}{2}\kappa^{-1}(\dot{a}^2+1)a & \text{closed}\end{array}\right. \end{aligned} \qquad (3)$$

which in the form of the constraint equation $H+H_{\mathrm{matter}}(a)=0$ with the matter Hamiltonian $H_{\mathrm{matter}}(a)$ yields the usual Friedmann equation

$$a^{-3}H+\rho_{\mathrm{matter}}(a)=0\,. \qquad (4)$$

In quantum geometry holonomies of the connection together with flux variables associated with the densitized triad are promoted to basic operators. One usually works in the connection representation where states are functionals on the infinite-dimensional space of connections via holonomies. After reducing to isotropic variables only one gauge invariant connection component c remains and states in the connection representation are functions of this single parameter. An orthonormal basis is given by the states

$$\langle c|n\rangle := \frac{\exp(\frac{1}{2}inc)}{\sqrt{2}\sin\frac{1}{2}c} \qquad (5)$$

labeled by an integer n. The states $|n\rangle$ are eigenstates of the basic derivative operator $\hat{p}$ which quantizes the isotropic triad component

$$\hat{p}|n\rangle = \tfrac{1}{6}\gamma l_{\mathrm{P}}^2 n|n\rangle \qquad (6)$$

where the Planck length $l_{\mathrm{P}}=\sqrt{\kappa\hbar}$ appears. We will later mainly use the volume operator which also has the states $|n\rangle$ as eigenstates:

$$\hat{V}|n\rangle = V_{\frac{1}{2}(|n|-1)}|n\rangle = (\tfrac{1}{6}\gamma l_{\mathrm{P}}^2)^{\frac{3}{2}}\sqrt{(|n|-1)|n|(|n|+1)}\,|n\rangle\,. \qquad (7)$$

Composite operators will be constructed from the volume operator together with multiplication operators

$$\begin{aligned} \cos(\tfrac{1}{2}c)|n\rangle &= \tfrac{1}{2}\left(\exp(\tfrac{1}{2}ic)+\exp(-\tfrac{1}{2}ic)\right)\frac{\exp(\frac{1}{2}inc)}{\sqrt{2}\sin(\frac{1}{2}c)} = \tfrac{1}{2}(|n+1\rangle+|n-1\rangle) & (8)\\ \sin(\tfrac{1}{2}c)|n\rangle &= -\tfrac{1}{2}i\left(\exp(\tfrac{1}{2}ic)-\exp(-\tfrac{1}{2}ic)\right)\frac{\exp(\frac{1}{2}inc)}{\sqrt{2}\sin(\frac{1}{2}c)} = -\tfrac{1}{2}i(|n+1\rangle-|n-1\rangle)\,. & (9)\end{aligned}$$

An example is presented by the inverse scale factor operator which is needed, e.g., in order to quantize the kinetic part of matter Hamiltonians. Since the triad and volume operators have a discrete spectrum containing the value zero, they do not have a densely defined inverse which would be necessary for a well-defined operator. However, there are methods in quantum geometry [20] which allow the classically divergent inverse scale factor to be turned into a well-defined operator [21]. To that end, one has to rewrite it classically as, e.g.,

$$a^{-1} = 6\gamma^{-1}\kappa^{-1}\{c, V^{\frac{1}{3}}\} = 12i\gamma^{-1}\kappa^{-1}e^{\frac{1}{2}ic}\{e^{-\frac{1}{2}ic}, V^{\frac{1}{3}}\}$$

using the symplectic structure. On the right hand side an inverse of the volume does not appear anymore, and it can easily be quantized by using the multiplication operators (8), (9), the volume operator (7), and turning the Poisson bracket into a commutator. As a result, the classical divergence of a^{-1}, which is also present in Wheeler–DeWitt quantizations where a simply becomes a multiplication operator, is cut off at small volume and the inverse scale factor operator is finite. We also note that rewriting a classical quantity in the way above introduces new possibilities for quantization ambiguities. For instance, we wrote the connection component c as an exponential which can appear with an arbitrary integer power. This integer determines at which volume the classical divergence is cut off [22], an effect which will be used and discussed in more detail later.

Another example for a composite operator which can be built from the volume and multiplication operators is the Hamiltonian constraint. We first recall the result in the flat isotropic case where methods used in the full theory [23] lead to the constraint operator [24, 10]

$$\hat{H}_{\mathrm{flat}} = 96i(\gamma^3\kappa l_{\mathrm{P}}^2)^{-1}\sin^2(\tfrac{1}{2}c)\cos^2(\tfrac{1}{2}c)\left(\sin(\tfrac{1}{2}c)\hat{V}\cos(\tfrac{1}{2}c) - \cos(\tfrac{1}{2}c)\hat{V}\sin(\tfrac{1}{2}c)\right) \qquad (10)$$

with action

$$\hat{H}_{\mathrm{flat}}|n\rangle = 3(\gamma^3\kappa l_{\mathrm{P}}^2)^{-1}\,\mathrm{sgn}(n)\left(V_{\frac{1}{2}|n|} - V_{\frac{1}{2}|n|-1}\right)(|n+4\rangle - 2|n\rangle + |n-4\rangle)\,. \qquad (11)$$

As usual, the constraint equation $\hat{H}|s\rangle = -\hat{H}_{\mathrm{matter}}|s\rangle$ turns into an evolution equation after transforming to a triad representation. In a triad representation the state $|s\rangle$ is represented by the coefficients $s_n(\phi)$ which appear in a decomposition of the state $|s\rangle = \sum_n s_n(\phi)|n\rangle$ in terms of triad eigenstates $|n\rangle$. Here, we also write the dependence of the wave function on a matter value ϕ which we do not need to specify further for our purposes. The matter Hamiltonian $\hat{H}_{\mathrm{matter}}$ acts on the ϕ-dependence of the state, but via metric components also on the gravitational part with label n. Since the triad has discrete spectrum in loop quantum cosmology, the evolution equation is a difference equation in the discrete internal time n, rather than a second order differential equation as in a Wheeler–DeWitt quantization. Still, for large volume $n \gg 1$ and small extrinsic curvature the difference equation can be well approximated by a differential equation such that the Wheeler–DeWitt approach and thus the semiclassical limit is reproduced at large volume [25]. At small volume close to the classical singularity, however, there are large corrections which must not be ignored.

In the flat model as discussed above, the intrinsic curvature is always zero such that small extrinsic curvature at large volume implies small total curvature and we expect almost classical behavior. In the closed model, on the other hand, the intrinsic curvature represented by the parameter $\Gamma = \frac{1}{2}$ is constant and never small. A general consistent loop quantization of homogeneous models [26] can be obtained by subtracting Γ from the connection component c such that we obtain the constraint operator

$$\hat{H} = 96i(\gamma^3\kappa l_{\mathrm{P}}^2)^{-1}\left(\sin^2\left(\tfrac{1}{2}(c-\tfrac{1}{2})\right)\cos^2\left(\tfrac{1}{2}(c-\tfrac{1}{2})\right)+\tfrac{1}{16}\gamma^2\right)\left(\sin(\tfrac{1}{2}c)\hat{V}\cos(\tfrac{1}{2}c)-\cos(\tfrac{1}{2}c)\hat{V}\sin(\tfrac{1}{2}c)\right) \tag{12}$$

with action

$$\hat{H}|n\rangle = 3(\gamma^3\kappa l_{\mathrm{P}}^2)^{-1}\,\mathrm{sgn}(n)\left(V_{\frac{1}{2}|n|}-V_{\frac{1}{2}|n|-1}\right)\left(e^{-i}|n+4\rangle-(2+\gamma^2)|n\rangle+e^{i}|n-4\rangle\right). \tag{13}$$

Transforming to the triad representation results in the difference equation

$$\begin{aligned} & \mathrm{sgn}(n+4)(V_{|n+4|/2}-V_{|n+4|/2-1})e^{i}s_{n+4}(\phi)-(2+\gamma^2)\,\mathrm{sgn}(n)(V_{|n|/2}-V_{|n|/2-1})s_n(\phi) \\ & +\mathrm{sgn}(n-4)(V_{|n-4|/2}-V_{|n-4|/2-1})e^{-i}s_{n-4}(\phi) \\ = \; & -\tfrac{1}{3}\gamma^3\kappa l_{\mathrm{P}}^2\hat{H}_{\mathrm{matter}}(n)s_n(\phi) \end{aligned} \tag{14}$$

where the reduced matter Hamiltonian $\hat{H}_{\mathrm{matter}}(n)$ defined by $\hat{H}_{\mathrm{matter}}|n\rangle\otimes|\phi\rangle =: |n\rangle\otimes\hat{H}_{\mathrm{matter}}(n)|\phi\rangle$ acts only on the ϕ-dependence of the wave function. One can easily check that it is possible to evolve through the classical singularity at $n=0$ in the same way as in the flat case [11]; thus, the singularity is absent in the loop quantization.

In order to derive a continuum approximation at large volume we have to find a continuous wave function $\psi(p,\phi)$ related to the discrete wave function $s_n(\phi)$ which does not vary strongly at small scales and solves an approximating differential equation derived from (14). Since a good continuum limit is a prerequisite for a correct classical limit, a wave function allowing such an interpolation is called pre-classical [12]. In the flat case the wave function s_n itself turned out to lead to the pre-classical solutions, while general considerations show that this cannot be the case in the presence of large intrinsic curvature [15]. In this case the wave function $\tilde{s} = \exp(3i\Gamma\hat{p}/\gamma l_{\mathrm{P}}^2)s$ with coefficients

$$\tilde{s}_n(\phi) = e^{\frac{1}{2}in\Gamma}s_n(\phi) = e^{\frac{1}{4}in}s_n(\phi) \tag{15}$$

where the phase factor cancels small-scale oscillations in s_n has to be used for a continuum limit. As in [25] for the flat case it can then be verified that the approximating differential equation for $n \gg 1$ is

$$\tfrac{1}{2}\left(\tfrac{4}{9}l_{\mathrm{P}}^4\frac{\partial^2}{\partial p^2}-1\right)\sqrt{|p|}\psi(p,\phi) = -\tfrac{1}{3}\kappa\hat{H}_{\mathrm{matter}}(p)\psi(p,\phi) \tag{16}$$

where $\psi(p,\phi)$ is an interpolation of the pre-classical $\tilde{s}_n(\phi)$. The left hand side can be written as $-\frac{1}{2}(4\hat{K}^2+1)a\psi(a,\phi)$ which shows that the Wheeler–DeWitt equation as a Schrödinger-like quantization of (4) is reproduced at large volume. The ordering is fixed

in (16) and follows from the non-singular difference equation of loop quantum cosmology. For de Sitter space with cosmological constant Λ we obtain the general solution

$$\sqrt{|p|}\psi(p) = A\mathrm{Ai}\left(-(\tfrac{3}{2}\Lambda)^{\frac{1}{3}}(p-\tfrac{3}{2}\Lambda^{-1})\right) + B\mathrm{Bi}\left(-(\tfrac{3}{2}\Lambda)^{\frac{1}{3}}(p-\tfrac{3}{2}\Lambda^{-1})\right)$$

in terms of Airy functions. The wave function $\psi(p)$ is either zero in $p=0$, if and only if $A\mathrm{Ai}\left((\tfrac{3}{2})^{\frac{4}{3}}\Lambda^{-\frac{2}{3}}\right) = -B\mathrm{Bi}\left((\tfrac{3}{2})^{\frac{4}{3}}\Lambda^{-\frac{2}{3}}\right)$, or diverges there.

3 Dynamical Initial Conditions

When we approach the classical singularity at $n=0$ by going to smaller values of n, we have to take into account discrepancies between the Wheeler–DeWitt equation and the difference equation of loop quantum cosmology. As a difference equation, the exact constraint equation (14) can be used as a recurrence relation which determines the wave function starting from initial values at some finite positive values of n. We can find values s_n for smaller n as long as the coefficient $V_{|n-4|/2} - V_{|n-4|/2-1}$ of lowest order in (14) does not vanish. However, it does vanish if and only if $n=4$ such that we cannot fix $s_0(\phi)$ in this way. As in [11] this does not mean that there is a singularity because we can evolve through $n=0$ and find all values of the wave function for negative internal time n. But the part of the constraint equation for $n=4$ has to be satisfied. Instead of fixing s_0 in terms of the initial data, it implies a linear relation between s_4 and s_8 which have already been determined in previous steps of the recurrence. Therefore, the constraint equation imposes implicitly a linear consistency condition on the initial data which serves as an initial condition. Since it is not imposed separately but follows from the evolution equation, it is called a dynamical initial condition [12]. Note that it only gives us partial information and does not fix the wave function completely in the presence of matter fields.

While the value s_0 of the wave function at the classical singularity is undetermined and it thus is impossible to formulate the dynamical initial conditions as conditions at $n=0$, they imply effectively that a pre-classical wave function has to approach zero close to $n=0$. In this sense, it is similar to DeWitt's initial condition which requires the wave function to vanish at the classical singularity. In contrast to that condition, however, the dynamical initial condition is well-posed in the discrete context (see, e.g., [14]).

For the closed model we can now also compare the dynamical initial condition with other traditional proposals, most importantly the no-boundary [3] and the tunneling proposal [4], which have been discussed only in this case. To do that we first consider the approximate differential equation for the Hamiltonian constraint (16) and rewrite it in terms of the scale factor a and in the "de Sitter" approximation where the scalar field potential is approximated by a (cosmological) constant V and the scalar field dependence is ignored

$$\left[a\frac{\mathrm{d}}{\mathrm{d}a}\frac{1}{a}\frac{\mathrm{d}}{\mathrm{d}a} - \frac{a^2}{l_{\mathrm{P}}^4}\left(9-6\kappa V a^2\right)\right] a\Psi(a) = 0\,. \tag{17}$$

This represents an ordering of the Hamiltonian constraint that has not previously been considered (compare, e.g., the general analysis in [27]). If we write $\tilde{\Psi} = a\Psi$ we see that in

order for Ψ to be bounded for small a we demand that $\tilde{\Psi}$ approach zero. For $\kappa a^2 V \ll 1$ the solution to (17) satisfying $\tilde{\Psi}(0) = 0$ is $AaI_{1/2}(\frac{3a^2}{2l_{\mathrm{P}}^2})$ where $I_{1/2}$ is the modified Bessel function and A is an arbitrary constant. When matched to the WKB solutions in the exponential region, this solution picks out the exponentially increasing WKB mode thus resembling the no-boundary wave function. We note however, that in the limit of small a Ψ approaches zero satisfying DeWitt's initial condition.

The previous analysis in terms of $\Psi(a)$ shares the same flaws of the traditional approach to quantum cosmology in that conditions on the wave function are specified near the classical singularity where we might expect to see large corrections due to the effects of discrete space. The evolution equation (14) needs to be solved in order to determine if the discrete effects modify the analysis. Since the dynamical initial condition forces the discrete wave function to approach zero for small a, the wave function increases for growing a before it starts to oscillate. Similarly to the previous analysis, the wave function exponentially increases in the classically forbidden region. A numerical solution to (14) with a cosmological constant exhibits this behavior (Fig. 1) reinforcing the previous analysis and indicating that the dynamical initial condition imposes conditions similar to the no-boundary proposal.

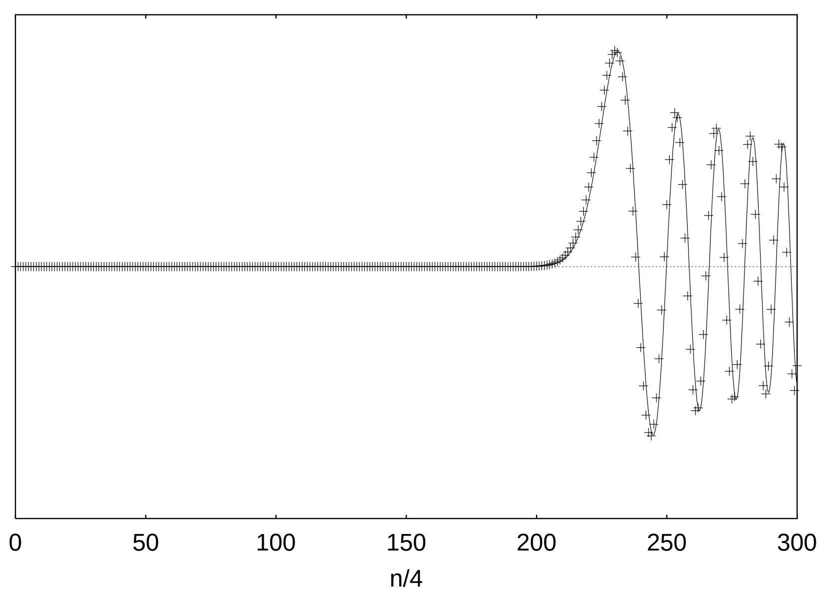

Figure 1: Pre-classical solution $(V_{|n|/2} - V_{|n|/2-1})\tilde{s}_n$ of the discrete equation (14) compared to a solution $\sqrt{p}\psi(p)$ of the Wheeler–DeWitt equation (16) such that ψ is regular at $p = 0$ (solid line) corresponding to de Sitter space with $\kappa\Lambda = 10^{-2} l_{\mathrm{P}}^{-2}$.

4 Inflation

It has been speculated that the no-boundary proposal does not predict large enough initial values for an inflaton field which suggests that a sufficient amount of inflation cannot be realized [28]. If correct, the same conclusion would apply for a wave function satisfying the dynamical initial condition. However, loop quantum cosmology presents an alternative mechanism of inflation [13] which does not necessarily require an inflaton field. This scenario exploits quantum modifications of the classical equations of motion implied by the discrete formulation. There are several types of corrections, but we can focus on the non-perturbative effect which results from a quantization of inverse metric components since it implies the most drastic changes. The effect contains an ambiguity parameter and by choosing it to be large the non-perturbative modification extends into the semiclassical regime. The wave function can then be approximated as a wave packet following the effective classical equations of motion which we will discuss now.

Eigenvalues of a density operator $\hat{d}_j = \widehat{a^{-3}}_j$, which quantizes the classical density a^{-3} in the kinetic part of a matter Hamiltonian, can be computed explicitly [22] for arbitrary values of the ambiguity parameter j which is a half-integer. Here we only need an approximate expression for the eigenvalues

$$d_{j,n} \sim (\tfrac{1}{6}\gamma l_{\mathrm{P}}^2 n)^{-\frac{3}{2}} p(n/2j)^6$$

where the function

$$p(q) = \tfrac{8}{77} q^{\frac{1}{4}} \left(7\left((q+1)^{\frac{11}{4}} - |q-1|^{\frac{11}{4}}\right) - 11q\left((q+1)^{\frac{7}{4}} - \mathrm{sgn}(q-1)|q-1|^{\frac{7}{4}}\right)\right) \qquad (18)$$

approaches one at large values but incorporates non-perturbative corrections for small $n < 2j$. In effective classical equations of motion the modified density will appear as a continuous function

$$d_j(a) = a^{-3} p(3a^2/\gamma l_{\mathrm{P}}^2 j)^6 \sim \left\{ \begin{array}{cl} \frac{12^6}{7^6}(\frac{1}{3}\gamma l_{\mathrm{P}}^2 j)^{-\frac{15}{2}} a^{12} & \text{for } a^2 \ll \frac{1}{3}\gamma l_{\mathrm{P}}^2 j \\ a^{-3} & \text{for } a^2 \gg \frac{1}{3}\gamma l_{\mathrm{P}}^2 j \end{array} \right. \qquad (19)$$

derived using $a^2 \sim \frac{1}{6}\gamma l_{\mathrm{P}}^2 n$. The precise form of the behavior for small a is subject to quantization ambiguities, but qualitatively it always has the above form. Additionally, the power of a in the small-a approximation is usually high as in the example used here.

In particular, $d_j(0) = 0$ and $d_j(a)$ increases as a function of a at small volume in contrast to the classical divergence. Consequently, the effective matter Hamiltonian, e.g., $H_{\mathrm{matter}}^{\mathrm{eff}}(a) = \frac{1}{2} d_{j_\phi}(a) p_\phi^2 + a^3 V(\phi)$ for a scalar, always satisfies

$$\lim_{a\to 0} a^{-1} H_{\mathrm{matter}}^{\mathrm{eff}}(a) = 0$$

even taking into account the kinetic term which would diverge classically. We can view the Friedmann equation $\dot{a}^2 + V(a) = 0$ as describing a classical motion in the effective potential

$$V(a) = 1 - \tfrac{2}{3}\kappa a^{-1} H_{\mathrm{matter}}^{\mathrm{eff}}(a) \qquad (20)$$

where we ignore the dependence on the matter field. Because the modified density is increasing at small a just like a potential term, the derivative $V'(a)$ of the effective potential is negative for small a. The effective classical equation of motion then implies

$$\ddot{a} = -\tfrac{1}{2}V'(a) > 0 \qquad \text{for small } a$$

such that the effective classical evolution at small a is inflationary. An inflaton field and therefore large initial values are not required for this kind of inflation which is purely due to quantum geometry effects.

It is, however, necessary that there be a large enough overlap between the region where the effective potential decreases and the classically allowed region where the effective potential is negative. In this intersection we can use the effective classical equations of motion and obtain an amount of inflation $a_{\mathrm{f}}/a_{\mathrm{i}}$ given by the initial scale factor a_{i} where the classically allowed region starts and the final scale factor a_{f} where the modified increasing density goes over into the standard density a^{-3}. For the flat model, the entire region of positive a was classically allowed, and a_{i} can be arbitrarily small, only restricted by the eventual breakdown of the effective classical approximation. In the closed model, however, the non-zero intrinsic curvature leads to a hill in the effective potential at small a such that the classically allowed region starts at a positive value of the scale factor. An approximate expression for the ratio $a_{\mathrm{f}}/a_{\mathrm{i}}$ can be derived in the case of a free massless scalar field. Using the small-a approximation of d_j in (19), the classical turning point corresponds to where the effective potential equals zero. This gives

$$a_{\mathrm{i}}^{11} \approx \frac{3}{\kappa}\frac{7^6}{12^6}\left(\frac{\gamma l_{\mathrm{P}}^2 j_\phi}{3}\right)^{15/2}\frac{1}{p_\phi^2} \tag{21}$$

where p_ϕ is the conjugate scalar field momentum. The inflationary regime ends where $d_{j_\phi}(a)$ takes on its maximum value which corresponds to $a_{\mathrm{f}} \approx \sqrt{\gamma l_{\mathrm{P}}^2 j_\phi/3}$ [22]. For practical values of the parameters the inflationary region is small and getting large amounts of inflation would require large values of p_ϕ.

4.1 Suppression of Intrinsic Curvature

With the quantization of the closed model presented so far it is hard to get a large amount of inflation because the intrinsic curvature introduces a potential hill which prohibits the scale factor from attaining very small values classically. The divergence of the kinetic part of matter Hamiltonians, which could cancel the potential hill, is cut off by quantum geometry effects. Since the inverse scale factor in isotropic cosmologies is related to the extrinsic curvature, this can also be interpreted as an extrinsic curvature cut-off. Because intrinsic and extrinsic curvature belong to the same geometrical object in a covariant treatment, one could expect that there is a similar effect which suppresses the intrinsic curvature represented by the spin connection $\Gamma = \frac{1}{2}$. This can in fact be realized by viewing the closed isotropic model as being embedded in the anisotropic Bianchi IX model. An anisotropic model has three independent triad components p^I which determine the intrinsic curvature

via the spin connection $\Gamma_1 = \frac{1}{2}(p^2/p^3 + p^3/p^2 - p^2p^3/(p^1)^2)$ and analogous components Γ_2 and Γ_3 (for Bianchi IX, [26]). The isotropic model can be obtained by fixing $p^1 = p^2 = p^3 = p$ such that $\Gamma_I = \frac{1}{2} = \Gamma$. The Hamiltonian constraint of the Bianchi IX model [15] requires a quantization of the spin connection which has to be a well-defined operator. Since the components Γ_I contain inverse powers of the triad, we have to use inverse triad operators introducing another ambiguity parameter j_Γ which can be different from the parameter j_ϕ used when quantizing the density $d = a^{-3}$ appearing in a scalar matter Hamiltonian. After reducing the quantized Bianchi IX spin connection to isotropy we obtain, with $p(q)$ of (18), eigenvalues

$$\Gamma_{j_\Gamma,n} = \tfrac{1}{2}\left(2p(n/2j_\Gamma)^2 - p(n/2j_\Gamma)^4\right)$$

replacing $\Gamma = \frac{1}{2}$ in the constraint equation. When n is small, the effective inverse triad components decrease such that Γ_{j_Γ} becomes smaller than $\frac{1}{2}$ for small volume and approaches zero at the classical singularity. For $n \gg 2j_\Gamma$ we have $\Gamma_{j_\Gamma,n} \sim \frac{1}{2}$. In fact, we do have a suppression of intrinsic curvature just as the extrinsic curvature is suppressed.

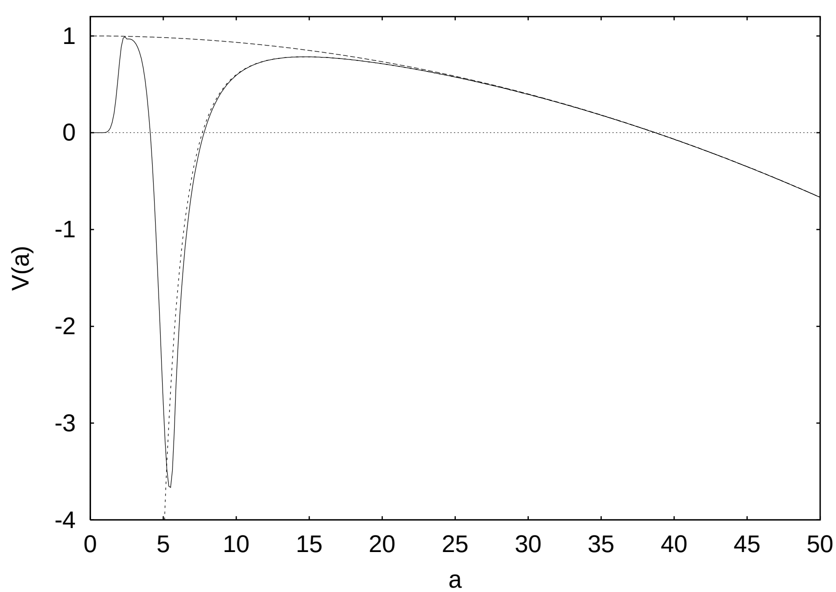

Figure 2: Example of an effective potential $V_\Gamma(a)$ (a in multiples of the Planck length) for a massless scalar with zero potential and a cosmological constant $\kappa\Lambda = 10^{-3} l_{\mathrm{P}}^{-2}$. The ambiguity parameters are $j_\Gamma = 20$, $j_\phi = 100$, and the scalar momentum is $\sqrt{\kappa} p_\phi = 100 l_{\mathrm{P}}^2$. For larger values of a the potential continues to decrease since the cosmological constant term dominates. The dashed lines are the effective potential without Γ-suppression and kinetic term (approaching one for $a = 0$), and without Γ-suppression and with a standard kinetic term (which diverges for $a = 0$).

As a consequence, the potential hill in the effective potential shrinks because the intrinsic curvature term which equaled one before is not constant anymore and approaches

zero: $\lim_{a\to 0} V_\Gamma(a) = 0$ rather than one (see Fig. 2), where

$$V_\Gamma(a) = 4\Gamma_{\rm eff}^2(a) - \tfrac{2}{3}\kappa a^{-1} H_{\rm matter}^{\rm eff}(a)$$

and

$$\Gamma_{\rm eff}(a) = \tfrac{1}{2}\left(2p(3a^2/\gamma l_{\rm P}^2 j_\Gamma)^2 - p(3a^2/\gamma l_{\rm P}^2 j_\Gamma)^4\right) \ .$$

The presence of a matter potential then leads to a small classically allowed region between $a = 0$ and some positive value (see Fig. 3) because the potential term increases faster than the suppression in the intrinsic curvature term owing to the large power in the small-a approximation (19). Initially, the effective potential decreases implying an inflationary epoch. After some time, the increasing Γ will start to dominate the effective potential which terminates inflation. When Γ increases such that the effective potential reaches positive values, the classically allowed region ends and a potential hill emerges which has to be tunneled through by the wave function. After the hill, we have the second phase of inflation already observed above which usually will only last for a small number of e-foldings. However, the first phase of inflation can start at values of the scale factor arbitrarily close to zero such that the amount of inflation can be very high just as in the flat case.

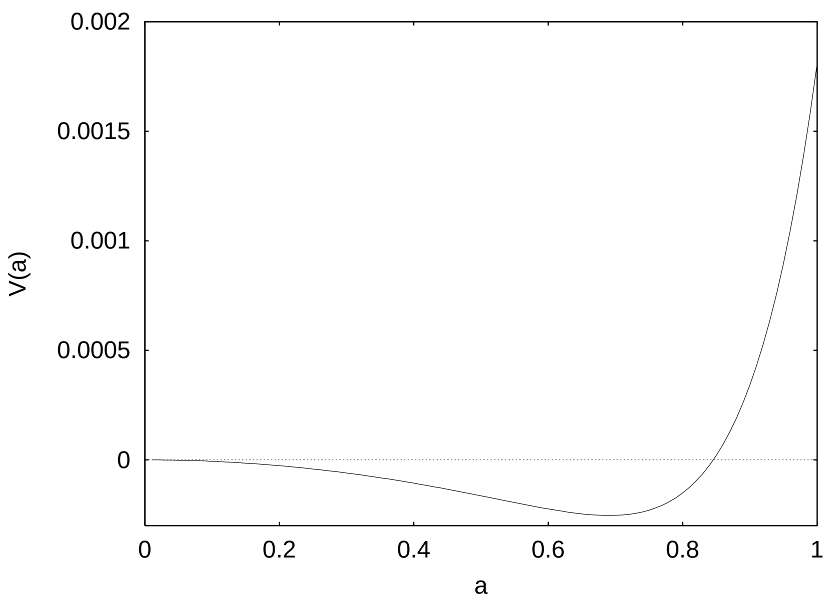

Figure 3: The effective potential with the same parameters as in Fig. 2 in the small-a classically allowed region.

So far we assumed that the ambiguity parameter j_Γ in the intrinsic curvature term is smaller than the parameter j_ϕ we had originally in matter terms. If this is not the case, the potential hill can even disappear completely if $j_\Gamma > j_\phi$ which implies a single phase of

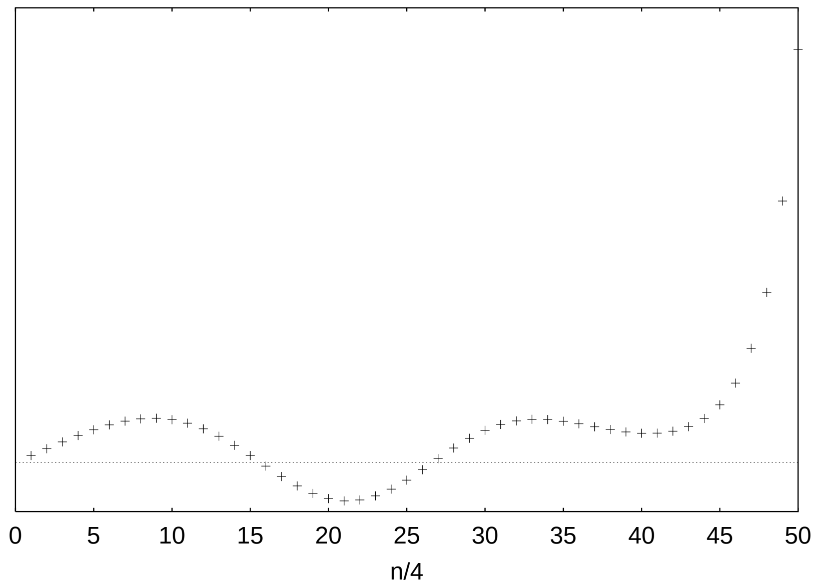

Figure 4: Wave function in the small-a classically allowed region with Γ-suppression for $j_\Gamma = 50$. (Obtained as a solution to equation (14) with $4\gamma^2\Gamma_{j_\Gamma,n}$ replacing γ^2 on the left hand side.)

inflation. The closed model is then very similar to the flat model for small volume due to the suppression of the intrinsic curvature term.

Another consequence of the suppression is that the wave function, which has to start with a small value at small n, oscillates in the first classically allowed region and can grow before it tunnels into the second region at large volume (Fig. 4). One could think that this scenario would lead to more similarities between the discrete wave function and that obtained with the tunneling proposal, which would be the case if the wave function decays in the classically forbidden region [28]. However, in the classically forbidden region we still have two independent solutions, one exponentially increasing and one exponentially decreasing. Since the dynamical initial condition, unlike the tunneling proposal, is not tailored to select the decreasing part, the increasing part will be present and dominate the solution (Fig. 5).

4.2 Horizon Problem

We now seek to answer quantitatively the question of whether or not enough inflation is obtained with the quantum modifications. A central problem with the Standard Big Bang (SBB) proposal is the 'horizon problem' in which the comoving region observed by the cosmological microwave background (CMB) is larger than the comoving forward light cone at the recombination time $t_{\rm rec}$ thus implying regions of the CMB out of causal contact.

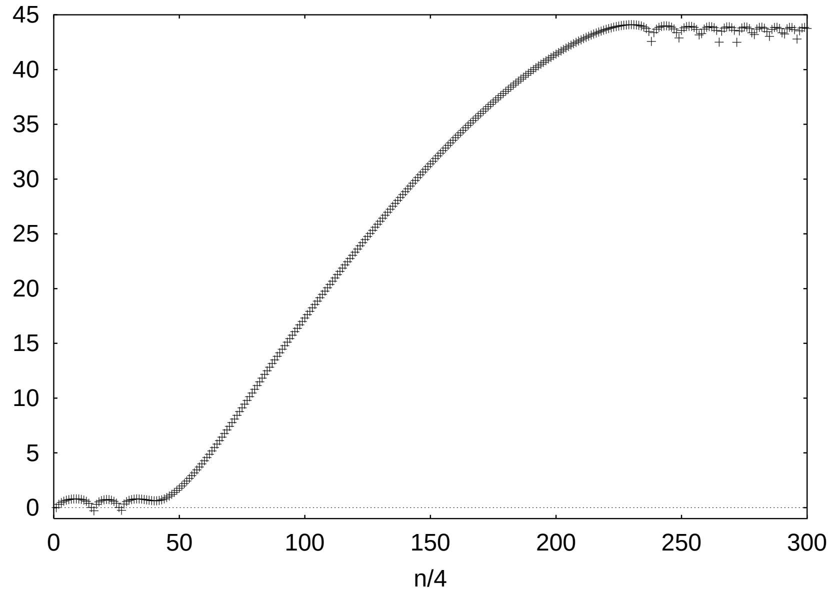

Figure 5: Logarithm of the absolute value of the wave function between the first two classically allowed regions ($j_\Gamma = 50$). In the classically forbidden region the wave function grows by more than 40 orders of magnitude and is very similar to a solution without Γ-suppression.

The forward light cone $l_{\mathrm{f}}(t_{\mathrm{rec}})$ is given by

$$l_{\mathrm{f}}(t_{\mathrm{rec}}) = \int_0^{t_{\mathrm{rec}}} \frac{1}{a(t)} \mathrm{d}t = \int_{a_{\mathrm{min}}}^{a_{\mathrm{rec}}} \frac{\mathrm{d}a}{a\dot{a}} \tag{22}$$

We consider matter in the form of a scalar field and the Friedmann equation becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\kappa d_{j_\phi}(a) p_\phi^2}{3a^3} + \frac{2\kappa}{3} V(\phi) + \frac{4(\Gamma^2 - \Gamma)}{a^2} \tag{23}$$

where $p_\phi = a^3\dot{\phi}$ is the conjugate momentum to the scalar field and Γ is the spin connection. We note that when $\Gamma = 1/2$ we regain the usual $-1/a^2$ term of the Friedmann equation for the closed model. For small a we can use the expansion for $d_j(a)$ in (19) to derive an approximate expression for $l_{\mathrm{f}}(t_{\mathrm{rec}})$.

First we consider a free, massless scalar field where the potential is zero and the conjugate momentum is constant. In the flat model the curvature term is zero and thus $\dot{a} \sim a^{11/2}$ and the forward light cone diverges as the integral is taken to small values of a. In the closed model, however, the integral begins at the minimum classical scale factor such that it does not diverge. Given that the amount of inflation is small for the closed model in this region, it would require very large values for p_ϕ to overcome the horizon problem. We will see that this problem is further evidenced in the flatness problem.

In the presence of a scalar potential, for small enough a the potential term will dominate over the kinetic term. We have shown that with Γ-suppression, the closed model behaves similarly to the flat model in the small a limit. Since the kinetic term is suppressed, the scalar field remains nearly constant and standard inflationary behavior occurs close to the classical singularity. There, a grows exponentially and $\dot{a} \sim a$, thus the forward light cone diverges beginning at the classical singularity.

4.3 Flatness Problem

The flatness problem represents itself as a fine-tuning problem. The dimensionless parameter $\Omega = \rho/\rho_{\rm c} = \kappa\rho/3H^2$, where $\rho_{\rm c} = 3H^2/\kappa$ is the critical density and $H = \dot{a}/a$ is the Hubble parameter, indicates the spatial topology: $\Omega > 1$ gives a closed universe whereas $\Omega \leq 1$ gives an open universe. In the SBB model, $\Omega = 1$ is an unstable fixed point; the deviation $\epsilon = \Omega - 1$ grows like a for a matter dominated universe and a^2 for a radiation dominated universe. Current measurements put a value of $\Omega \approx 1$ [29] requiring Ω to be extraordinarily close to one in the early universe in the SBB scenario.

We consider $\epsilon = \Omega - 1 = k/\dot{a}^2$ and use the Friedmann equation (23) to write ϵ as a function of the scale factor a

$$\epsilon = k\left(\tfrac{1}{3}\kappa d_{j_\phi}(a)a^{-1}p_\phi^2 + \tfrac{2}{3}\kappa a^2 V(\phi) - k\right)^{-1}. \tag{24}$$

In standard inflation, the universe undergoes a period of exponential growth in which ϵ is driven close to zero (spatial flatness) thus avoiding the necessity of fine-tuning Ω and predicting a current value of Ω close to one for most models. With quantum modifications ϵ is also driven toward zero during the inflationary period. Considering a massless scalar field, for small a the kinetic term is an increasing function of a and thus ϵ decreases until $d_{j_\phi}(a)$ becomes a decreasing function after which ϵ grows with a. The minimum value of ϵ can be approximated by using the fact that $d_{j_\phi}(a)$ takes its maximum value at $a_{\rm peak} \approx \sqrt{\gamma l_{\rm P}^2 j_\phi/3}$ with $d_{j,\max} \approx a_{\rm peak}^{-3}$ [22]. This gives a minimum value

$$\epsilon_{\min} \approx \left(\frac{\kappa p_\phi^2}{3}\left(\frac{3}{\gamma l_{\rm P}^2 j_\phi}\right)^2 - 1\right)^{-1}. \tag{25}$$

Assuming there exists a region where the matter kinetic term dominates the curvature term we find that $\epsilon_{\min} \sim j_\phi^2/p_\phi^2$, thus in order to drive ϵ sufficiently close to zero we need small values of j_ϕ and large values of p_ϕ. This presents a problem since j_ϕ needs to be sufficiently large so that the quantum corrections occur in the region where the classical equations of motion are valid. If we take a reasonable value of $j_\phi = 100$ we find that in order for ϵ to have the required accuracy of 10^{-60} near the Plank regime (assuming no second round of standard inflation), $\sqrt{\kappa}p_\phi$ needs to be on the order of $10^{30} l_{\rm P}^2$. It thus seems unlikely that the quantum corrections alone can provide enough inflation for the closed model.

The inclusion of a potential does not help the matter. To drive ϵ close enough to zero would require either unnaturally large values of the potential or very large values of the

final scale factor at the end of inflation. The latter scenario would then be equivalent to standard inflation.

4.4 Summary

We have seen that the ratio $a_{\rm f}/a_{\rm i}$ cannot be large enough for a sufficient amount of inflation with natural values of the different parameters, which are mainly the half-integer ambiguity label j_ϕ and the initial value p_ϕ of the scalar momentum. As a function of these parameters, $a_{\rm f}/a_{\rm i}$ increases only slowly with p_ϕ and decreases with j_ϕ (small values of j_ϕ also appear more natural from a conceptual point of view). If we invoke Γ-suppression introducing a new parameter j_Γ, the situation looks better since we have an early inflationary region with small $a_{\rm i}$. However, in this regime the viability of the effective framework remains to be understood.

The best scenario can be obtained by using quantum geometry inflation in order to generate standard inflation. This happens naturally whenever there is a matter component which is well approximated by a scalar with a flat potential. In the quantum geometry regime the equations of motion of the scalar have a different form than usually because of the modified density. Equations of motion can be derived from the Hamiltonian which for a scalar has the effective form

$$H_{\rm matter}^{\rm eff}(a) = \frac{1}{2} d_{j_\phi}(a) p_\phi^2 + a^3 V(\phi)\,.$$

This yields

$$\dot\phi = \{\phi, H_{\rm matter}^{\rm eff}(a)\} = d_{j_\phi}(a) p_\phi$$

and

$$\dot p_\phi = \{p_\phi, H_{\rm matter}^{\rm eff}(a)\} = -a^3 V'(\phi)$$

resulting in the second order equation of motion

$$\ddot\phi = p_\phi \frac{{\rm d}d_{j_\phi}(a)}{{\rm d}t} + d_{j_\phi}(a)\dot p_\phi = p_\phi\, \dot a\, d'_{j_\phi}(a) - a^3 d_{j_\phi}(a) V'(\phi) = a\frac{{\rm d}\log d_{j_\phi}(a)}{{\rm d}a}\, H\dot\phi - a^3 d_{j_\phi}(a) V'(\phi)$$

with the Hubble parameter H. In the standard case $d_j(a) = a^{-3}$, we have $a\,{\rm d}\log d_j(a)/{\rm d}a = -3$ and the first order term serves as a friction which leads to slow roll for a sufficiently flat potential. For the modified $d_j(a)$ at small volume, however, $d_j(a)$ increases and thus $a\,{\rm d}\log d_j(a)/{\rm d}a$ is positive. In this case, the friction term has the opposite sign and forces the scalar to move up the potential (see Fig. 6, whose initial values have been chosen for the purpose of illustration; one can easily achieve larger maximal values of ϕ by using a larger initial $\dot\phi$).

Thus, quantum geometry modifications drive the scalar up the potential during early phases, even if it starts in a minimum, thereby producing large initial values for standard inflation. It remains to be seen whether the standard inflationary phase will wash away any possible signature of the quantum geometry phase, or if it can be distinguished from other scenarios.

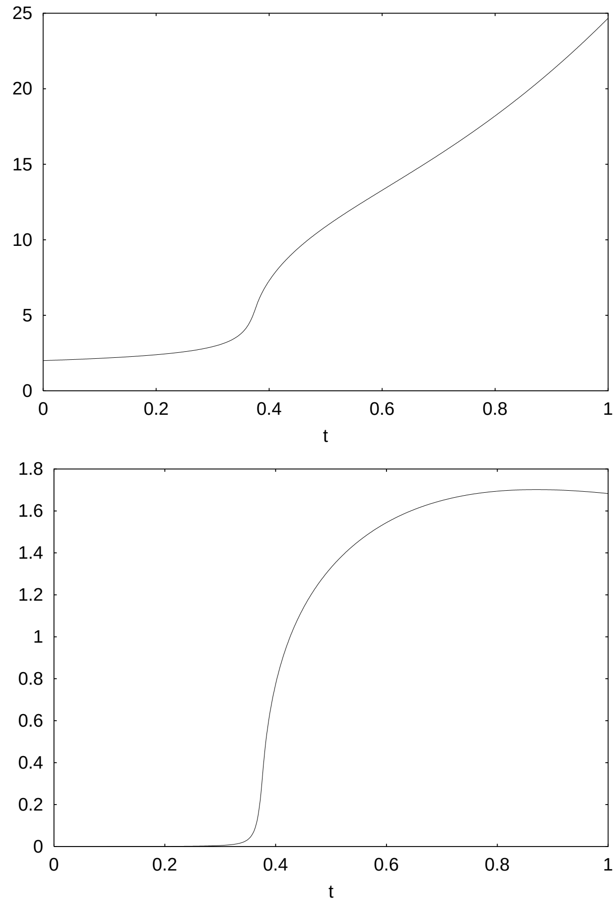

Figure 6: Scale factor a (top) and scalar ϕ (bottom) in Planck units with a mass term $\kappa V(\phi) = 10^{-3}\hbar\phi^2/2$. The first phase of quantum geometry inflation (which ends at $t \approx 0.4$) leads to large initial ϕ for a second phase of slow roll inflation (with $j_\phi = 100$ and initial values $\phi_0 = 0$, $\sqrt{\kappa}\dot{\phi}_0 \approx 10^{-5} l_{\mathrm{P}}^{-1}$ at $a_0 = 2l_{\mathrm{P}}$ such that $\sqrt{\kappa} p_{\phi,0} \approx 100 l_{\mathrm{P}}^2$).

5 Discussion

We have studied the closed isotropic model in loop quantum cosmology and seen that it reproduces all of the main general results: its evolution is non-singular and it predicts dynamical initial conditions as well as the occurrence of inflation. The advantage of using the closed model is that we are able to compare with older attempts to understand the quantum dynamics. In fact, those approaches are reproduced at large volume, but many new properties result at small volume. The initial conditions, for instance, have a very different origin since they are derived in part from the dynamical law and not imposed by hand. Furthermore, the factor ordering of the constraint is fixed by the requirement of a non-singular evolution. Older attempts have often been plagued by the factor ordering problem because different orderings can lead to qualitatively different results and even

invalidate certain proposals [27]. Still, as in any quantization there are many ambiguities which can be included with certain parameters like j_ϕ. These ambiguities, however, usually lead only to quantitative differences leaving the main results unchanged. They can, therefore, be used for a phenomenological analysis.

As for dynamical initial conditions, the constraint equation requires the wave function to approach very small values at the classical singularity such that the effect is often similar to DeWitt's initial condition. However, due to effects of the discreteness the initial value problem is well-posed (which presents some realization of the Planck potential of [30]). The first quantization we used for the closed model is closely related to the no-boundary proposal while it would be different from the tunneling proposal (we note, however, that we ignored the kinetic term of the matter Hamiltonian in (17) as usual in this context; in a more general discussion one can also expect more differences to the no-boundary proposal due to quantum geometry modifications of the kinetic term as in [13]). However, this quantization does not take into account a possible suppression of intrinsic curvature which can be realized by embedding the isotropic model in the anisotropic Bianchi IX model. This introduces a second ambiguity parameter j_Γ which influences the results: If $j_\phi < j_\Gamma$, there is no potential hill and the whole region of scale factors between zero and a possible recollapse value is allowed classically. In this case, the closed model is very similar to the flat model at small volume. Otherwise, there will be a potential hill and we have two classically allowed regions. For small j_Γ, the first region is only small and the wave function again resembles that of the no-boundary proposal. If j_Γ is increased, the first classically allowed region grows such that the wave function can increase there before it tunnels into the second classically allowed region. However, the result remains closer to the no-boundary proposal than to the tunneling proposal because the exponentially increasing branch of the wave function will dominate in the classically forbidden region. Still, depending on the values of ambiguity parameters we obtain modifications of the old proposals realizing different aspects of them. Furthermore, the values will influence the amount of inflation achieved within a given model.

We also have to give some cautionary remarks: While the quantization of the flat model can be regarded as being very close to that of the full theory, the closed model requires a special input due to the large intrinsic curvature. In the flat model, the intrinsic curvature vanishes identically, and in the full theory it can be ignored locally. Therefore, it has to be verified that the physical effects are not artifacts of the special techniques but indeed model the full theory. As we have seen here, viewing a symmetric model as being embedded in a less symmetric one can lead to important effects. In our case, those effects (Γ-suppression) were welcome and improved the model from the point of view of cosmological model building. It is certainly a very important issue to investigate a similar, but much more complicated, correspondence between symmetric models and the full theory. Also more complicated models contain many ambiguity parameters whose physical role and reasonable values are not completely clear yet. Still, they provide a rich ground for phenomenology which allows studying the same objects and effects present in the full theory in a much more simplified context.

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